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CAN AN INFINITE LEFT-PRODUCT OF NONNEGATIVE MATRICES BE EXPRESSED IN TERMS OF INFINITE LEFT-PRODUCTS OF STOCHASTIC ONES?

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Abstract. If a left-product $M_n \dots M_1$ of square complex matrices converges to a nonnull limit when $n \to \infty$ and if the M_n belong to a finite set, it is clear that there exists an integer n_0 such that the M_n, $n \ge n_0$, have a common right-eigenvector V for the eigenvalue 1. Now suppose that the M_n are nonnegative and that V has positive entries. Denoting by Δ the diagonal matrix whose diagonal entries are the entries of V, the stochastic matrices $S_n = \Delta^{-1} M_n \Delta$ satisfy $M_n \dots M_{n_0} = \Delta S_n \dots S_{n_0} \Delta^{-1}$, so the problem of the convergence of $M_n \dots M_1$ reduces to the one of $S_n \dots S_{n_0}$. In this paper we still suppose that the M_n are nonnegative but we do not suppose that V has positive entries. The first section details the case of the 2×2 matrices, and the last gives a first approach in the case of $d \times d$ matrices.

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Introduction

The problem of the convergence of $M_1 \dots M_n$ or $M_n \dots M_1$ for all the sequences (M_n) with terms in a finite set of complex matrices, is studied for instance in [1], [2] and [3]. The same problem in case of stochastic matrices is also classical, see for instance [9, chapter 4]. On the other side there exist much results about the distribution of the product matrix $M_1 \dots M_n$ where the M_i are taken in a set of stochastic matrices endowed with some probability measure. In [6] and [7], Mukherjea, Nakassis and Ratti give conditions for the limit distribution of $M_1 \dots M_n$ to be discrete or continuous singular; this contains for instance the case of the Erdős measure [4]. It is well known that in much cases the normalized product $M_1 \dots M_n$, if the M_i are taken in a finite set of nonnegative matrices endowed with some positive probability P, converges P-almost everywhere to a rank one matrix; but this general result should be more consistent if one can to specify the P-negligeable set of divergence.

Key words and phrases. LCP sets, RCP sets, products of nonnegative matrices, products of stochastic matrices.

Notice that the Erdős measure is studied more in detail in [8] by an other method, using a finite set of matrices and the asymptotic properties of the columns in the products of matrices taken in this set.

In the present paper we first consider (§1) the left-products and the right-products of 2×2 matrices (resp. 2×2 stochastic matrices). For the left-products of stochastic matrices, the hypothesis that the matrices belong to a finite set is not necessary. We recover the known results by giving several formulations of the necessary and sufficient conditions of convergence.

In §2 we associate to any sequence (M_n) of nonnegative $d\times d$ matrices, some sequences of stochastic ones, let $\left(S_n^{(i)}\right)$ for $1\le i\le t$. The convergence of $M_n\dots M_1$ is equivalent to the one of the $S_n^{(i)}\dots S_1^{(i)}$ and some additional condition.

1. Products of 2×2 matrices

1.1. **Convergent left-products.** Suppose that the left-product $M_n\dots M_1$ of some nonnegative matrices $M_n=\begin{pmatrix} a_n & b_n\\ c_n & d_n\end{pmatrix}$ converges to a nonnull limit, let Q, and that the set $\{M\ ;\ \exists n, M=M_n\}$ is finite. It is clear that from a rank n_0, the M_n belong to the set

$$\{M\ ;\ \text{ there exists infinitely many } n \text{ such that } M=M_n\}$$

and consequently, for any M in this set, $MQ=M$. In other words the nonnull columns of Q are eigenvectors – for the eigenvalue 1 – of any M_n, $n\ge n_0$. If for instance this eigenvector is $\begin{pmatrix} v_1\\ 0\end{pmatrix}$, then $M_n=\begin{pmatrix} 1 & b_n\\ 0 & d_n\end{pmatrix}$ and $M_n\dots M_{n_0}=\begin{pmatrix} 1 & \sum_{i=n_0}^n b_id_{i-1}\dots d_{n_0}\\ 0 & d_n\dots d_{n_0}\end{pmatrix}$

Proposition 1.1. *$M_n\dots M_{n_0}$ converges to a nonnull limit, for any fixed n_0 and when $n\to\infty$, if and only if*

- *either the M_n have from a certain rank, a common right-eigenvector $\begin{pmatrix} v_1\\ v_2\end{pmatrix}$ for the eigenvalue 1, with $v_1v_2>0$, and the left-product of the stochastic matrices $S_n=\begin{pmatrix} \frac{1}{v_1} & 0\\ 0 & \frac{1}{v_2}\end{pmatrix}M_n\begin{pmatrix} v_1 & 0\\ 0 & v_2\end{pmatrix}$ from any rank n_0 converges;*
- *or the M_n have from a certain rank $\begin{pmatrix} 1\\ 0\end{pmatrix}$ for common positive right-eigenvector with respect to the eigenvalue 1, the sum $\displaystyle\sum_{n=n_0}^{\infty} b_nd_{n-1}\dots d_{n_0}$ is finite and $d_n\dots d_{n_0}$ converges, for any n_0, when $n\to\infty$;*

- *or the M_n have from a certain rank $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for common positive right-eigenvector with respect to the eigenvalue 1, the sum $\sum_{n=n_0}^{\infty} c_n a_{n-1} \dots a_{n_0}$ is finite and $a_n \dots a_{n_0}$ converges, for any n_0, when $n \to \infty$.*

1.2. **Case of stochastic matrices.** The case of the matrices

$$S_n = \begin{pmatrix} x_n & 1-x_n \\ y_n & 1-y_n \end{pmatrix}, \; x_n, y_n \in [0,1],$$

is also trivial because one can compute the left-product

$$Q_n := S_n \dots S_1 = \begin{pmatrix} t_n & 1-t_n \\ s_n & 1-s_n \end{pmatrix} \quad \text{where} \quad \begin{cases} s_n := \sum_{i=1}^n y_i \det Q_{i-1} \\ t_n := s_n + \det Q_n \\ Q_0 := I. \end{cases} \tag{1}$$

To find the conditions for the sequence of matrices (Q_n) to converge one can use the relation

$$s_n = s_{n_0-1} + s_{n_0,n} \det Q_{n_0-1}, \quad \text{where } s_{n_0,n} := \sum_{i=n_0}^{n} \left(y_i \prod_{n_0 \le j < i} \det S_j \right). \tag{2}$$

$s_{n_0,n}$ belongs to $[0,1]$ because it is one of the entries of the stochastic matrix $S_n \dots S_{n_0}$. Hence, in case $\det Q_n$ has limit 0 the relation (2) implies that (s_n) is Cauchy; so (s_n), (t_n) and (Q_n) converge.
Suppose now that $\det Q_n$ do not have limit 0. Since $\det Q_n = \prod_{i=1}^n (x_i - y_i)$ with $x_i - y_i \in [-1,1]$, the non-increasing sequence $(|\det Q_n|)$ has a positive limit δ hence $|x_n - y_n|$ has limit 1; (x_n, y_n) cannot have other limit points than $(0,1)$ and $(1,0)$.
In case $(0,1)$ is one of its limit points, $y_n|\det Q_{n-1}|$ do not tend to 0 hence the series $\sum_n y_n \det Q_{n-1}$ diverges and (Q_n) also do.
In case $(1,0)$ is the unique limit point of (x_n, y_n), $x_n - y_n$ is positive from a rank n_0. Since the series $\sum_n \log|x_n - y_n|$ converges to $\log \delta$, the inequalities $\sum_{n \ge n_0} \log(x_n - y_n) \le \sum_{n \ge n_0} \log(1 - y_n) \le -\sum_{n \ge n_0} y_n$ prove that the series $\sum_n y_n$ converges. Since $\det Q_n$ has limit δ or $-\delta$ according to the sign of $\det Q_{n_0-1}$, the sequences (s_n), (t_n) and (Q_n) converge.

Consider now the right-product $P_n := S_1 \dots S_n$ and suppose that the S_n belong to a finite set. As noticed in [3] it is clear that the nonnull rows of the limit matrix P, if this matrix exists, are nonnegative left-eigenvectors – for the eigenvalue 1 – of each matrix S such that $S_n = S$ for infinitely many n, because the equality $P_n = P_{n-1} S$ implies $P = PS$. So P_n can converge only if the S_n, for n greater or equal to some integer n_0, have a common nonnegative left-eigenvector for the eigenvalue 1.

We suppose there exists such a left-eigenvector, let L, and we search the condition for (P_n) to converge. Notice that the S_n for $n \geq n_0$ commute: since any 2×2 stochastic matrix S has left-eigenvector $\begin{pmatrix} 1 & -1 \end{pmatrix}$ for the eigenvalue $\det S$, both vectors L and $\begin{pmatrix} 1 & -1 \end{pmatrix}$ are orthogonal to the columns of $S_nS_{n'} - S_{n'}S_n$ and consequently this matrix is null. So we fall again in the case of the left-products.

In case $S_{n_0}\dots S_n$ diverges, nevertheless the sequence of the row-vectors $LS_{n_0}\dots S_n$ converges (it is constant). Let L' be some row-eigenvector not colinear to L; considering the invertible matrix M whose rows are L and L', $MS_{n_0}\dots S_n$ obviously diverges hence $L'S_{n_0}\dots S_n$ also do. Consequently $S_1\dots S_n$ diverges if and only if at least one of the rows of $S_1\dots S_{n_0-1}$ is not colinear to L.

We have proved the following

Proposition 1.2. *Let (S_n) be a sequence of 2×2 stochastic matrices, namely*

$$S_n = \begin{pmatrix} x_n & 1-x_n \\ y_n & 1-y_n \end{pmatrix}.$$

(i) The left-product $Q_n = S_n\dots S_1$ converges if and only if $\prod_{k=1}^n (x_k - y_k)$ has limit 0 or (x_n, y_n) has limit $(1,0)$ when $n\to+\infty$.

(ii) It diverges only in the case where $\sum_n (1-|x_n-y_n|)$ converges and S_n do not tend to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(iii) If it converges, $\lim_{n\to\infty} Q_n = \begin{pmatrix} s+q & 1-s-q \\ s & 1-s \end{pmatrix}$ where $s := \sum_n y_n \det Q_{n-1}$ and $q := \lim_{n\to\infty}(\det Q_n)$.

(iv) Suppose now that the S_n belong to a finite set. Then the right-product $P_n = S_1\dots S_n$ converges if and only if there exists n_0 such that

_ the matrices S_n for $n\geq n_0$ have a common left-eigenvector for the eigenvalue 1

_ and either $\lim_{n\to\infty}\prod_{k=n_0}^n (x_k-y_k) = 0$, or $\lim_{n\to\infty}(x_n,y_n) = (1,0)$, or the rows of $S_1\dots S_{n_0-1}$ are colinear to the left-eigenvector.

(v) Suppose the S_n, $n\geq 1$, have a common left-eigenvector with respect to the eigenvalue 1. Then the S_n commute and $P_n = Q_n$.

2. Left products of $d\times d$ nonnegative matrices

Let us first give one example in order to illustrate the proposition that follows: we consider the products $Q_n = M_n\dots M_1$, where the M_i belong to the set of nonnegative matrices of the form

$$M=\begin{pmatrix} a & b & 3-3a-2b & e & 4-4a-3b-5e & 1 & 0\\ c & d & 2-3c-2d & f & 3-4c-3d-5f & 0 & 1\\ 0 & 0 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & g & 5-5g & 0 & 1\\ 0 & 0 & 0 & h & 1-5h & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & x & 1-x\\ 0 & 0 & 0 & 0 & 0 & y & \frac{1}{2}-y \end{pmatrix}$$

where $a, b, c, d, e, f, g, h, x, y$ are some reals such that M has exactly twenty four nonnull entries. Since the eigenspace associated to the eigenvalue 1 is generated by $\begin{pmatrix}3\\2\\1\\0\\0\\0\\0\end{pmatrix}$ and $\begin{pmatrix}4\\3\\0\\5\\1\\0\\0\end{pmatrix}$, we can associate to each matrix M two submatrices with positive eigenvectors: the submatrix $M^{\{1,2,3\}}$ of the entries of M with row and column indexes in $\{1,2,3\}$ and the submatrix $M^{\{1,2,4,5\}}$ of the entries of M with row and column indexes in $\{1,2,4,5\}$. Then we associate two stochastic matrices $S=\Delta^{-1}M^{\{1,2,3\}}\Delta$ and $S'=\Delta'^{-1}M^{\{1,2,3\}}\Delta'$, where $\Delta=\begin{pmatrix}3&0&0\\0&2&0\\0&0&1\end{pmatrix}$ and $\Delta'=\begin{pmatrix}4&0&0&0\\0&3&0&0\\0&0&5&0\\0&0&0&1\end{pmatrix}$; namely

$$S=\begin{pmatrix} a & 2b/3 & 1-a-2b/3\\ 3c/2 & d & 1-3c/2-d\\ 0 & 0 & 1\end{pmatrix} \quad\text{and}\quad S'=\begin{pmatrix} a & 3b/4 & 5e/4 & 1-a-3b/4-5e/4\\ 4c/3 & d & 5f/3 & 1-4c/3-d-5f/3\\ 0 & 0 & g & 1-g\\ 0 & 0 & 5h & 1-5h\end{pmatrix}.$$

Now we obtain 21 of the 49 entries of $Q_n = M_n \dots M_1$ in function of two products of stochastic matrices, and the other entries in the first five columns of Q_n are null: indeed Q_n has for submatrices $M_n^{\{1,2,3\}}\dots M_1^{\{1,2,3\}} = \Delta S_n \dots S_1 \Delta^{-1}$ and $M_n^{\{1,2,4,5\}}\dots M_1^{\{1,2,4,5\}} = \Delta' S'_n \dots S'_1 \Delta'^{-1}$.

The products $S_n \dots S_1$ and $S'_n \dots S'_1$ converge to rank 1 matrices: use for instance [5], or use the previous section and the formula for the products of triangular-by-blocks matrices that is,

$$\prod_{i=n}^{1} \begin{pmatrix} A_i & B_i \\ 0 & D_i \end{pmatrix} = \begin{pmatrix} \prod_{i=n}^{1} A_i & \sum_{i=1}^{n} A_n \dots A_{i+1} B_i D_{i-1} \dots D_1 \\ 0 & \prod_{i=n}^{1} D_i \end{pmatrix}.$$

The limit of Q_n is a rank 2 matrix of the form $\begin{pmatrix} 0 & 0 & 3 & 4\alpha & 4\beta & 4\gamma+6 & 4\delta+6 \\ 0 & 0 & 2 & 3\alpha & 3\beta & 3\gamma+4 & 3\delta+4 \\ 0 & 0 & 1 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 5\alpha & 5\beta & 5\gamma & 5\delta \\ 0 & 0 & 0 & \alpha & \beta & \gamma & \delta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

The following proposition generalizes what we have seen on the example. For any $d \times d$ matrix M and for any $K, K' \subset \{1, \dots, d\}$ we denote by M^K the submatrix of the entries of M whose row index and column index belong to K and by $M^{K,K'}$ the submatrix of the entries of M whose row index belongs to K and column index belongs to K'. By commodity we use the same notation for the row-matrices L (resp. the column-matrices V): L^K (resp. V^K) is the row-matrice (resp. column-matrice) of the entries of L (resp. V) whose index belong to K.

Proposition 2.1. *Let (M_n) be a sequence of nonnegative $d \times d$ matrices that belong to a given finite set. The left-product $Q_{n,n_0} := M_n \dots M_{n_0}$ converges to a nonnull limit when $n \to \infty$ – for each positive integer n_0 – if and only if (M_n) satisfies both conditions:*

(i) there exist some subsets of $\{1, \dots, d\}$, let $K_1, \dots, K_t$ with complementaries $K_1^c, \dots, K_t^c$, and some diagonal matrices with positive diagonals, let $\Delta_1, \dots, \Delta_t$, such that the $S_n^{(i)} = \Delta_i^{-1} M_n^{K_i} \Delta_i$ are stochastic, the $M_n^{K_i^c, K_i}$ are null, and $\lim_{n\to\infty} S_n^{(i)} \dots S_{n_0}^{(i)}$ exists for any i and n_0;

(ii) setting $K = \cup_i K_i$, $\lim_{n\to\infty} M_n^{K^c} \dots M_{n_0}^{K^c}$ is the null matrix for any n_0 and the series $\sum_{i=n_0}^{\infty} M_n^K \dots M_{i+1}^K M_i^{K,K^c} M_{i-1}^{K^c} \dots M_{n_0}^{K^c}$ converges.

Proof. If the conditions (i) and (ii) are satisfied, the entries of $M_n \dots M_{n_0}$ with column index in K converges either to 0 or to the entries of the matrices $\lim_{n\to\infty} \Delta_i S_n^{(i)} \dots S_{n_0}^{(i)} \Delta_i^{-1}$, $i = 1, \dots, t$. Consequently $M_n \dots M_{n_0}$ converges, by using the formula of product of triangular-by-blocs matrices:

$$\prod_{i=n}^{n_0}\begin{pmatrix} A_i & B_i \\ 0 & D_i \end{pmatrix} = \begin{pmatrix} \prod_{i=n}^{n_0} A_i & \sum_{i=n_0}^{n} A_n \dots A_{i+1} B_i D_{i-1} \dots D_{n_0} \\ 0 & \prod_{i=n}^{n_0} D_i \end{pmatrix}.$$

Conversely suppose that $Q_{n,n_0} = M_n \dots M_{n_0}$ converges to a nonnull limit for any $n_0 \in \mathbb{N}$. Since the M_n belong to a finite set, we can choose n_0 large enough such that all the matrices M that are equal to M_n for at least one $n \geq n_0$, are also equal to M_n for infinitely many n. Then the nonnull columns of the limit matrix Q are right-eigenvectors of all the M_n, $n \geq n_0$, for the eigenvalue 1, because the equality $Q_{n,n_0} = M_n Q_{n-1,n_0} = M Q_{n-1,n_0}$ implies $Q = MQ$. Let t be the rank of Q, $t = 0$ if Q is null, and denote by $V_1, \dots, V_t$ the linearly independent columns of Q. We denote also by K_i the set of the indexes of the nonnull entries in V_i, and by Δ_i the diagonal matrix whose diagonal entries are the nonnull entries of V_i. Now (i) results from the equality $M_n^{D,K_i} V_i^{K_i} = V_i$, $D := \{1, \dots, d\}$; $S_n^{(i)} \dots S_{n_0}^{(i)}$ converges because $\Delta_i S_n^{(i)} \dots S_{n_0}^{(i)} \Delta_i^{-1}$ is a submatrix of Q_{n,n_0}.
On the other side, denoting by K the union of the K_i, the product $M_n^{K^c} \dots M_{n_0}^{K^c}$ and the sum of products $\sum_{i=n_0}^{n} M_n^K \dots M_{i+1}^K M_i^{K,K^c} M_{i-1}^{K^c} \dots M_{n_0}^{K^c}$ converge when $n \to \infty$ because they are submatrices of Q_{n,n_0}. The first converges to 0: by the definitions of K and the vectors V_i, the rows of Q whose indexes belong to K^c are null.

□

Remark 2.1. The condition that $\sum_{i=1}^{\infty} M_n^K \dots M_{i+1}^K M_i^{K,K^c} M_{i-1}^{K^c} \dots M_1^{K^c}$ converges cannot be avoided. Suppose for instance that $M_n^K \dots M_1^K$ converges and that M_n^{K,K^c} is for any n the identity $\frac{d}{2} \times \frac{d}{2}$ matrix, let $I_{\frac{d}{2}}$, d even. Suppose also $M_n^{K^c} = d_n I_{\frac{d}{2}}$ for any n, where the positive reals d_n satisfy $\lim_{n\to\infty} d_n \dots d_1 = 0$ and $\sum_{n=1}^{\infty} d_n \dots d_1 = \infty$. Then if the M_n have a common right-eigenvector for the eigenvalue 1, it has the form $\begin{pmatrix} W \\ \theta_{\frac{d}{2}} \end{pmatrix}$ where W is an eigenvector of the M_n^K and $\theta_{\frac{d}{2}}$ is the $\frac{d}{2}$-dimensional null column-vector. Since $\left(\sum_{i=1}^{n} M_n^K \dots M_{i+1}^K M_i^{K,K^c} M_{i-1}^{K^c} \dots M_1^{K^c}\right) W = \left(\sum_{i=1}^{n} d_{i-1} \dots d_1\right) W$ diverges, $M_n \dots M_1$ also do although $M_n^{K^c} \dots M_1^{K^c}$ converges to the null matrix.

Remark 2.2. The two conditions of (ii) are satisfied – assuming that the ones of (i) are – if all the submatrices $M_n^{K^c}$ have spectral radius less than 1, or if their eigenvalues greater or equal to 1 disappear in the product $M_{n+h}^{K^c} \dots M_n^{K^c}$ for some fixed h and for any n.

Remark 2.3. Let us compare now the problem of the convergence of $Q_n = M_n \dots M_1$ to the one of

$$R_n := \frac{M_n \dots M_1}{\|M_n \dots M_1\|}.$$

Let (M_n) be a sequence of complex-valued matrices such that R_n converges. Since the limit matrix R has norm 1, it has some nonnull columns; let us prove that they are right-eigenvectors of each matrix M that occurs infinitely many times in the sequence (M_n). The nonnegative real $\lambda_n := \|MR_{n-1}\|$ is bounded by $\|M\|$ and satisfy $MR_{n-1} = \lambda_n R_n$ for any n such that $M_n = M$, so it has at least one limit point λ that satisfy $MR = \lambda R$, and the columns of R are right-eigenvectors of M for the eigenvalue λ.

Suppose that $\lambda \neq 0$ – perhaps it is not possible that $\lambda = 0$. The convergence of R_n can hold without the convergence of $\left(\frac{1}{\lambda}M_n\right) \dots \left(\frac{1}{\lambda}M_1\right)$, see for instance the case where all the M_n are equal to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. But conversely the convergence of this last product to a nonnull matrix, for some $\lambda \neq 0$, implies obviously the one of R_n.

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